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Deformed and extended Galilei group Hopf algebras

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Abstract. The $\hat{\kappa}$ -deformed extended Galilei Hopf group algebra, $\text{Fun}_{\hat{\kappa}}(\tilde{G}_{(m)})$, is introduced. It provides an explicit example of a deformed group with cocycle bi-crossproduct structure, and is shown to be the contraction limit of a pseudoextension of the κ -Poincaré group algebra. The possibility of obtaining other deformed extended Galilei groups is discussed, including one obtained from a non-standard Poincaré deformation.

1. Introduction

In a recent paper [1] we have discussed the ‘non-relativistic’ contractions of the deformed κ -Poincaré algebra [2], using the pseudoextension mechanism [3, 4] to search for an *extended* deformed Galilei algebra. In short, this process explains how a trivial (direct product) extension by the phase group may lead by contraction to a non-trivial central extension or, in other words, how a two-coboundary may generate a non-trivial two-cocycle by contraction. In the Lie algebra case, the pseudoextension mechanism explains, for instance, how the direct product $\mathcal{P} \times u(1)$, where \mathcal{P} is the Poincaré algebra, may lead to the extended Galilei algebra $\tilde{\mathcal{G}}_{(m)}$; other interesting examples may be given both in the undeformed and the deformed case [5].

A result of the analysis in [1] is that there are two possible contractions of the κ -Poincaré algebra [2] $\mathcal{P}_{\kappa} \equiv \mathcal{U}_{\kappa}(\mathcal{P})$ depending on how the constant c is hidden in κ , since the standard $c \rightarrow \infty$ limit (κ unaltered) leads to the *undeformed* Galilei Hopf algebra $\mathcal{U}(\mathcal{G})$.

(a) $\mathcal{U}_{\tilde{\kappa}}(\mathcal{G})$. If the deformation parameter κ (which has dimensions of inverse length, $[\kappa] = L^{-1}$) is replaced by $\tilde{\kappa}/c$ in \mathcal{P}_{κ} ($[\tilde{\kappa}] = T^{-1}$), the usual redefinitions ($P_i = X_i$, $P_0 \equiv X_t/c$, $N_i = cV_i$ with $[X_t] = T^{-1}$, $[X] = L^{-1}$, $[V] = L^{-1}T$) in $\mathcal{U}_{\kappa}(\mathcal{P})$ lead in the $c \rightarrow \infty$ contraction limit to the Hopf algebra $\mathcal{G}_{\tilde{\kappa}} \equiv \mathcal{U}_{\tilde{\kappa}}(\mathcal{G})$ given by

$$\begin{aligned}
 [J_i, J_j] &= \epsilon_{ijk} J_k & [J_i, X_j] &= \epsilon_{ijk} X_k & [J_i, X_t] &= 0 \\
 [J_i, V_j] &= \epsilon_{ijk} V_k & [X_i, X_t] &= 0 & [X_i, X_j] &= 0 \\
 [V_i, X_t] &= X_i & [V_i, X_j] &= \delta_{ij} \frac{1}{2\tilde{\kappa}} X^2 - \frac{1}{\tilde{\kappa}} X_i X_j & [V_i, V_j] &= 0 \\
 \Delta X_t &= X_t \otimes 1 + 1 \otimes X_t & \Delta X_i &= X_i \otimes 1 + \exp(-X_t/\tilde{\kappa}) \otimes X_i & & (1.1) \\
 \Delta J_i &= J_i \otimes 1 + 1 \otimes J_i & \Delta V_i &= V_i \otimes 1 + \exp(-X_t/\tilde{\kappa}) \otimes V_i + \frac{\epsilon_{ijk}}{\tilde{\kappa}} X_j \otimes J_k \\
 S(X_t) &= -X_t & S(X_i) &= -\exp(X_t/\tilde{\kappa}) X_i \\
 S(J_i) &= -J_i & S(V_i) &= -\exp(X_t/\tilde{\kappa}) V_i + \frac{1}{\tilde{\kappa}} \epsilon_{ijk} \exp(X_t/\tilde{\kappa}) X_j J_k
 \end{aligned}$$

$\epsilon(\text{all}) = 0$. This algebra (found in [6] in another basis) has been shown to have [1] a bi-crossproduct [7] structure: $\mathcal{U}_{\hat{\kappa}}(\mathcal{G}) = \mathcal{U}(R \circ B) \bowtie \mathcal{U}_{\hat{\kappa}}(\mathcal{T}r_4) = \mathcal{H} \bowtie \mathcal{A}$ so that $\mathcal{A} = \mathcal{U}_{\hat{\kappa}}(\mathcal{T}r_4)$ is the time (X_t) and space (X_i) translations Hopf subalgebra of $\mathcal{U}_{\hat{\kappa}}(\mathcal{G})$ and $\mathcal{H} = \mathcal{U}(R \circ B)$ is the undeformed Hopf algebra generated by the rotations (J_i) and boosts (V_i) (here and in the rest of the paper, we shall use the generic notation $\mathcal{K} = \mathcal{H} \bowtie \mathcal{A}$ to denote the (right \bowtie / left \bowleftarrow) bi-crossproduct structure [7] of \mathcal{K}).

(b) $\mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{G}}_{(m)})$. If κ is replaced by $\hat{\kappa}c$ ($[\hat{\kappa}] = L^{-2}T$), the κ -Poincaré algebra $\mathcal{U}_{\kappa}(\mathcal{P})$ still leads by contraction to the undeformed Galilei Hopf algebra $\mathcal{U}(\mathcal{G})$. However, this redefinition of the deformation parameter allows us to obtain a non-trivial deformation if the contraction is now performed on a pseudoextension of \mathcal{P}_{κ} , $\mathcal{U}_{\kappa}(\mathcal{P}) \times \mathcal{U}(u(1))$. The result [1] is the $\hat{\kappa}$ -deformed extended Galilei algebra $\tilde{\mathcal{G}}_{(m)\hat{\kappa}} \equiv \mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{G}}_{(m)})$, where the mass parameter m is introduced, as in the undeformed case, through the two-coboundary defining the pseudoextension. Denoting by Ξ the additional (eleventh) central generator, the Hopf structure of $\mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{G}}_{(m)})$ is given by the commutators of the undeformed Hopf algebra $\mathcal{U}(\tilde{\mathcal{G}}_{(m)})$, primitive coproducts etc, but for the exceptions given below:

$$\begin{aligned} [V_i, X_j] &= \delta_{ij} \frac{\hat{\kappa}}{2} (1 - \exp(2m\Xi/\hat{\kappa})) & \Delta X_i &= X_i \otimes 1 + \exp(m\Xi/\hat{\kappa}) \otimes X_i \\ \Delta V_i &= V_i \otimes 1 + \exp(m\Xi/\hat{\kappa}) \otimes V_i & S(X_i) &= -\exp(-m\Xi/\hat{\kappa}) X_i \\ S(V_i) &= -\exp(-m\Xi/\hat{\kappa}) V_i. \end{aligned} \quad (1.2)$$

The additional generator Ξ has the dimensions of inverse on an action; nevertheless it may be rendered dimensionless (in a quantum context) multiplying it by the Planck constant \hbar . The Casimir operators for $\mathcal{U}_{\hat{\kappa}}(\mathcal{G})$ and $\mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{G}}_{(m)})$ were also found in [1]. Since setting $\hat{\kappa} = 0$ in (1.2) the undeformed Hopf algebra structure of the enveloping algebra $\mathcal{U}(\tilde{\mathcal{G}}_{(m)})$ is recovered (and, in particular, $[V_i, X_j] = -m\delta_{ij}\Xi$), we see that the deformation enters in (1.2) only through the central generator. In fact, it may be shown [1, 8] that $\mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{G}}_{(m)})$ has the (right–left) cocycle bi-crossproduct [7] structure $\mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{G}}_{(m)}) = \mathcal{U}(\mathcal{G})^{\psi} \bowtie_{\xi_{(m)}} \mathcal{U}(u(1))$, where the right $\mathcal{U}(\mathcal{G})$ -module action (\bowtie) α (within the generic notation $\mathcal{H}^{\psi} \bowtie_{\xi} \mathcal{A}$, $\alpha(a \otimes h) \equiv a \triangleleft h = a \epsilon_{\mathcal{H}}(h)$) of $\mathcal{G}_{(m)}$ on $u(1)$ is taken to be trivial since Ξ is central in $\mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{G}}_{(m)})$ and the left $u(1)$ -comodule coaction (\bowleftarrow) β is given by

$$\beta(X_i) = \exp(m\Xi/\hat{\kappa}) \otimes X_i \quad \beta(V_i) = \exp(m\Xi/\hat{\kappa}) \otimes V_i \quad (1.3)$$

(β is trivial on X_t, J_i , i.e. $\beta(X_t) = 1 \otimes X_t, \beta(J_i) = 1 \otimes J_i$). The cocycle $\xi_{(m)}$ may be taken to satisfy†

$$\xi_{(m)}(V_i, X_j) - \xi_{(m)}(X_j, V_i) = \delta_{ij} \frac{\hat{\kappa}}{2} \left(1 - \exp\left(\frac{2m\Xi}{\hat{\kappa}}\right) \right) \quad (1.4)$$

and, finally, $\psi : \mathcal{U}(\mathcal{G}) \rightarrow \mathcal{U}(u(1)) \otimes \mathcal{U}(u(1))$ is trivial (for $\mathcal{H}^{\psi} \bowtie_{\xi} \mathcal{A}$, triviality means $\psi(h) = 1_{\mathcal{A}} \otimes 1_{\mathcal{A}\epsilon}(h)$). We note that the above structure is not the only one possible. It may be seen, for example, that $\mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{G}}_{(m)})$ is also the bi-crossproduct $\mathcal{H} \bowleftarrow \mathcal{A}$ of the commutative Hopf subalgebra $\mathcal{A} = \mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{T}}r_4)$ of $\mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{G}}_{(m)})$ generated by $\langle X_t, X_i, \Xi \rangle$ and the undeformed Hopf algebra $\mathcal{H} = \mathcal{U}(R \circ B)$ generated by $\langle V_i, J_i \rangle$, with right action $\alpha : \mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{T}}r_4) \otimes \mathcal{U}(R \circ B) \rightarrow \mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{T}}r_4)$ and left coaction $\beta : \mathcal{U}(R \circ B) \rightarrow \mathcal{U}_{\hat{\kappa}}(\tilde{\mathcal{T}}r_4) \otimes \mathcal{U}(R \circ B)$

† In [1, 8] $\xi_{(m)}$ was taken to be antisymmetric (cf (1.4)) but we wish to be less restrictive here (this freedom is related with the election of coboundary as we shall discuss in the next section).

given by

$$\begin{aligned} X_t \triangleleft V_i &= -X_i & X_i \triangleleft V_j &= -\delta_{ij} \frac{\hat{\kappa}}{2} (1 - \exp(2m \Xi / \hat{\kappa})) & \Xi \triangleleft V_i &= 0 \\ X_t \triangleleft J_i &= 0 & X_i \triangleleft J_j &= \epsilon_{ijk} J_k & \Xi \triangleleft J_i &= 0 & \beta(V_i) &= \exp(m \Xi / \hat{\kappa}) \otimes V_i \\ \beta(J_i) &= 1 \otimes J_i. \end{aligned} \quad (1.5)$$

The existence of more than one structure for a deformed Hopf algebra, as is the case for an ordinary Lie algebra, is not uncommon; see [5].

Deformed Newtonian and enlarged Newtonian spacetimes may be introduced by looking at the duals $\tilde{\mathcal{N}}_{\hat{\kappa}}$ and $\tilde{\mathcal{N}}_{\tilde{\kappa}}$ of the commutative Hopf subalgebras $\mathcal{U}_{\hat{\kappa}}(\mathcal{T}r_A)$, $\mathcal{U}_{\tilde{\kappa}}(\tilde{\mathcal{T}}r_A)$ of $\mathcal{U}_{\hat{\kappa}}(\mathcal{G})$ [(1.1)] and $\mathcal{U}_{\tilde{\kappa}}(\tilde{\mathcal{G}}_{(m)})$ generated by $\langle X_t, X_i \rangle$ and $\langle X_t, X_i, \Xi \rangle$ respectively. If a differential calculus covariant under rotations and boosts is now introduced, it turns out [1] that the full commutativity for the contraction diagrams, which involve the differential calculus [9] on the κ -Minkowski space [10–12], is obtained only for the extended Newtonian spacetime $\tilde{\mathcal{N}}_{\tilde{\kappa}}$. Although the addition of a (non-invariant) one-form to $\Gamma(\tilde{\mathcal{N}}_{\tilde{\kappa}})$ allows us to solve the rotations/boosts covariance equations, this new form cannot be interpreted (unlike in $\Gamma(\tilde{\mathcal{N}}_{\tilde{\kappa}})$) as the differential of an additional variable. This, as the previous discussion, poses the question of whether it is possible to find an extended deformed Galilei algebra for the $\tilde{\kappa}$ -deformation as is the case for $\mathcal{U}_{\tilde{\kappa}}(\tilde{\mathcal{G}}_{(m)})$. At the same time, it is convenient to have the dual or ‘group-like’ expressions ($\text{Fun}_{\tilde{\kappa}}(G)$, $\text{Fun}_{\tilde{\kappa}}(\tilde{\mathcal{G}}_{(m)})$) of the above deformed algebras and to look at the closure of the contraction diagrams involving the deformed Poincaré Hopf group algebra $\text{Fun}_{\tilde{\kappa}}(P)$. We shall address and answer these questions here, including for completeness the treatment of the non-standard deformations of the Poincaré group algebra in $(1+1)$ dimensions and its Galilean contractions. Besides describing possible deformed Galilean groups, our discussion will also provide some examples of deformed groups of the cocycle bi-crossproduct type, a structure of which very few [13] examples are known, especially from the group (rather than algebra) point of view.

2. The extended deformed Galilei group $\text{Fun}_{\tilde{\kappa}}(\tilde{\mathcal{G}}_{(m)})$

Since $\mathcal{U}_{\tilde{\kappa}}(\tilde{\mathcal{G}}_{(m)})$ has a cocycle bi-crossproduct [7] structure, its dual $\text{Fun}_{\tilde{\kappa}}(\tilde{\mathcal{G}}_{(m)})$ also has this structure. In fact, the dual $K = H_{\tilde{\xi}} \blacktriangleright \bar{\psi} A$ of $\mathcal{K} = \mathcal{H}^{\psi} \blacktriangleleft_{\tilde{\xi}} \mathcal{A}$, where H and A are the duals of \mathcal{H} and \mathcal{A} , is determined by the dual operations $(\bar{\beta}, \bar{\alpha}, \bar{\psi}, \bar{\xi})$ of $(\alpha, \beta, \xi, \psi)$. Quite often the deformation properties of a deformed Hopf algebra \mathcal{K} with the generic structure $\mathcal{K} = \mathcal{H}^{\psi} \blacktriangleleft_{\tilde{\xi}} \mathcal{A}$ are mostly described by those among the mappings $\alpha : \mathcal{A} \otimes \mathcal{H} \rightarrow \mathcal{A}$, $\beta : \mathcal{H} \rightarrow \mathcal{A} \otimes \mathcal{H}$, $\xi : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{A}$, $\psi : \mathcal{H} \rightarrow \mathcal{A} \otimes \mathcal{A}$ which involve the deformation parameter. As a result, the dual deformed Hopf group algebra $K = H_{\tilde{\xi}} \blacktriangleright \bar{\psi} A$ may be found in an easier way by looking for the respective dual maps $(\bar{\beta} : A \rightarrow A \otimes H, \bar{\alpha} : A \otimes H \rightarrow H (\bar{\alpha} : a \otimes h \mapsto a \bar{\triangleright} h), \bar{\psi} : A \rightarrow H \otimes H \text{ and } \bar{\xi} : A \otimes A \rightarrow H)$ than by direct computation or from ‘quasi-classical’ bialgebra structure considerations, since H and A are either undeformed or known. In these cases, the dualization process may be performed in individually simpler steps.

Since the cocycle bi-crossproduct structure of the deformed extended Galilei algebra $\mathcal{U}_{\tilde{\kappa}}(\tilde{\mathcal{G}}_{(m)})$ was found for α and ψ trivial, $\bar{\beta}$ and $\bar{\xi}$ are also trivial, and only the duals $\bar{\alpha}$ and $\bar{\psi}$ of β (equation (1.3)) and ξ (equation (1.4)), respectively, need to be determined since $H = \text{Fun}(G)$ and $A = \text{Fun}(U(1))$ are undeformed. Since the rotation part in the Galilei group is not important in the discussion, we shall restrict ourselves initially to one spatial

dimension. The commutative, non-cocommutative Hopf algebra structure of $\text{Fun}(G(1+1))$ is obvious:

$$\begin{aligned} [t, v] = 0 \quad [x, v] = 0 \quad [t, x] = 0 \\ \Delta t = 1 \otimes t + t \otimes 1 \quad \Delta x = 1 \otimes x + x \otimes 1 - t \otimes v \quad \Delta v = v \otimes 1 + 1 \otimes v \quad (2.1) \\ S(t) = -t \quad S(x) = -x - vt \quad S(v) = -v \quad \epsilon(t, x, v) = 0 \end{aligned}$$

where the group algebra generators t (x) correspond to the time (space) translation and v to the boost. The duality relations are

$$\langle X, x \rangle = 1 \quad \langle X_t, t \rangle = 1 \quad \langle V, v \rangle = 1 \quad (2.2)$$

the others being zero. To determine $\text{Fun}_{\hat{\kappa}}(\tilde{G}_{(m)})$ we introduce an additional generator χ dual to Ξ , $\langle \Xi, \chi \rangle = 1$, with $\Delta \chi = \chi \otimes 1 + 1 \otimes \chi$, $S(\chi) = -\chi$, $\epsilon(\chi) = 0$. Then, using equation (1.3) and relations such as $\langle V, \chi \bar{\triangleright} v \rangle \equiv \langle V, \tilde{\alpha}(\chi \otimes v) \rangle = \langle \beta(V), \chi \otimes v \rangle$, the left action $\tilde{\alpha}$ ($\bar{\triangleright}$) dual to β is immediately seen to be

$$\chi \bar{\triangleright} t = 0 \quad \chi \bar{\triangleright} x = \frac{m}{\hat{\kappa}} x \quad \chi \bar{\triangleright} v = \frac{m}{\hat{\kappa}} v. \quad (2.3)$$

Finding the dual cocycle $\bar{\psi}$ of ξ requires a more careful analysis. Since, as mentioned, explicit examples [13, 1, 5] of cocycle bi-crossproduct constructions for deformed Hopf algebras are rather scarce, we shall provide it with some detail. Recalling that $\xi : \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}) \rightarrow \mathcal{U}(u(1))$, let us take

$$\xi(V, X) = B(\Xi) \quad \xi(X, V) = B(\Xi) - \frac{\hat{\kappa}}{2} \left(1 - \exp\left(\frac{2m\Xi}{\hat{\kappa}}\right) \right) \quad (2.4)$$

(so that (1.4) is satisfied), where $B(\Xi)$ is as yet undetermined. Since $[V, X_t] = X$, it follows that

$$\begin{aligned} \xi(V, VX_t) - \xi(V, X_t V) = B(\Xi) \\ \xi(VX_t, V) - \xi(X_t V, V) = B(\Xi) - \frac{\hat{\kappa}}{2} \left(1 - \exp\left(\frac{2m\Xi}{\hat{\kappa}}\right) \right). \end{aligned} \quad (2.5)$$

We may now use the two-cocycle condition [14, 7] on ξ to obtain more information. Since the right action α (\triangleleft) was trivial, this condition reduces to

$$\xi(h_{(1)}g_{(1)} \otimes f)\xi(h_{(2)} \otimes g_{(2)}) = \xi(h \otimes g_{(1)}f_{(1)})\xi(g_{(2)} \otimes f_{(2)}) \quad (2.6)$$

where $h, g, f \in \mathcal{U}(\mathcal{G})$. Assuming the natural normalization $\xi(h \otimes 1) = \xi(1 \otimes h) = 1\epsilon(h)$ and $\xi(1 \otimes 1) = 1$, the application of (2.6) to the Galilei algebra generators, which have primitive coproducts, reduces it to $\xi(hg \otimes f) = \xi(h \otimes gf)$. Thus, we have, in particular,

$$\xi(VX_t, V) = \xi(V, X_t V) \equiv A(\Xi) \quad (2.7)$$

introducing a new unknown function. Then, with

$$\langle B(\Xi), \chi \rangle = B \quad \langle A(\Xi), \chi \rangle = A \quad (2.8)$$

we find, besides equations (2.5) and (2.7), the relations

$$\begin{aligned} \xi(V, VX_t) = A(\Xi) + B(\Xi) = \xi(V^2, X_t) \\ \xi(X_t, V^2) = A(\Xi) - B(\Xi) + \frac{\hat{\kappa}}{2} \left(1 - \exp\left(\frac{2m\Xi}{\hat{\kappa}}\right) \right). \end{aligned} \quad (2.9)$$

Now, using that

$$\left\langle \frac{\hat{\kappa}}{2} \left(1 - \exp\left(\frac{2m\Xi}{\hat{\kappa}}\right) \right), \chi \right\rangle = -m \quad \langle VX_t, vt \rangle = 1 \quad \langle V^2, v^2 \rangle = 2$$

the dual $\bar{\psi}$ of ξ is found to be

$$\begin{aligned} \bar{\psi}(\chi) = & A[\frac{1}{2}v^2 \otimes t + vt \otimes v + v \otimes vt + \frac{1}{2}t \otimes v^2] + B[v \otimes x + x \otimes v + v \otimes vt \\ & + \frac{1}{2}v^2 \otimes t - \frac{1}{2}t \otimes v^2] + m[x \otimes v - \frac{1}{2}t \otimes v^2]. \end{aligned} \tag{2.10}$$

The last bracket is recognized as a form of the Galilei non-trivial two-cocycle [15]; since the first two depend on the constants A and B , they must correspond to two-coboundaries. This may be trivially checked in a Lie group language if we think of the terms on the right-hand side of (2.10) as the product of the unprimed and primed group parameters defining the standard group two-cocycle, $\omega(g, g')$ say. In this way, it is seen that the first two brackets are generated by the one-cochains $\eta(g) = -\frac{1}{2}v^2t$ and $\eta(g) = -\frac{1}{2}v^2t - vx$, respectively, through $\omega_{\text{cob}}(g, g') = \eta(g) + \eta(g') - \eta(gg')$. Thus, we need not worry here about the cocycle condition for $\bar{\psi}$, although we shall come back to it in section 3. We shall only note now that, in the present Hopf algebra context, the condition expressing that $\bar{\psi}$ is a coboundary is translated into

$$\begin{aligned} \bar{\psi}_{\text{cob}}(\chi) = & 1 \otimes \gamma(\chi) + \gamma(\chi^{(1)}) \otimes \chi^{(2)} - \Delta(\gamma(\chi)) \\ = & 1 \otimes \gamma(\chi) + \gamma(\chi) \otimes 1 - \Delta(\gamma(\chi)) \end{aligned} \tag{2.11}$$

where γ is a linear mapping $\gamma : A \rightarrow H$ which is convolution invertible (i.e. there exists γ^{-1} such that $\gamma(a_{(1)})\gamma^{-1}(a_{(2)}) = \gamma^{-1}(a_{(1)})\gamma(a_{(2)}) = \epsilon(a)$); $\gamma(\chi)$ is given by $-\frac{1}{2}v^2t$ [$-\frac{1}{2}v^2t - vx$] for the first [second] bracket in (2.10). Ignoring the coboundaries, we thus find that $\text{Fun}_{\hat{\kappa}}(\tilde{G}_{(m)})$ is defined by equations (2.1) to which one has to add those dictated by (2.3), i.e.

$$[\chi, t] = 0 \quad [\chi, x] = \frac{m}{\hat{\kappa}}x \quad [\chi, v] = \frac{m}{\hat{\kappa}}v \tag{2.12}$$

the two cocycle (2.10), the last term of which modifies the primitive coproduct $\Delta\chi$ to read

$$\Delta\chi = \chi \otimes 1 + 1 \otimes \chi + m(x \otimes v - \frac{1}{2}t \otimes v^2) \tag{2.13}$$

and the antipode and counit of χ ,

$$S(\chi) = -\chi + m(xv + \frac{1}{2}tv^2) \quad \epsilon(\chi) = 0. \tag{2.14}$$

As expected from $\mathcal{U}_{\hat{\kappa}}(\tilde{G}_{(m)})$, the non-commutative nature of $\text{Fun}_{\hat{\kappa}}(\tilde{G}_{(m)})$ only shows up in the commutation properties (2.12) of the additional generator χ .

Moving now to (1 + 3) dimensions by including the rotations is not difficult. The only modified coproducts and antipodes are

$$\begin{aligned} \Delta x_i = & 1 \otimes x_i + x_j \otimes R^j_i - t \otimes v_i & \Delta v_i = & 1 \otimes v_i + v_j \otimes R^j_i \\ S(x_i) = & -x_j(R^{-1})^j_i - tv_j(R^{-1})^j_i & S(v_i) = & -v_j(R^{-1})^j_i \end{aligned} \tag{2.15}$$

and equations such as (2.2) and (2.12)–(2.14) acquire the appropriate vector indices (for instance, $S(\chi) = -\chi + m[xv + \frac{1}{2}tv^2]$.) As for R^j_i , it commutes with all other group algebra generators, $(\Delta R)^i_j = R^i_k \otimes R^k_j$, and its presence modifies the cocycle to read

$$\bar{\psi}(\chi) = m(x_i \otimes R^i_j v^j - \frac{1}{2}t \otimes v^2). \tag{2.16}$$

If the angle-like generator $\varphi = \chi/\hbar$ is used, $\text{Fun}_{\hat{\kappa}}(\tilde{G}_{(m)})$ may be rightly called the *deformed quantum-mechanical Galilei group*.

3. $\text{Fun}_{\hat{\kappa}}(\tilde{G}_{(m)})$ as a contraction of a pseudoextension of $P_{\hat{\kappa}}$ ($\text{Fun}_{\hat{\kappa}}(P)$)

Consider the κ -Poincaré group algebra $P_{\kappa} \equiv \text{Fun}_{\kappa}(P)$ in $(1+1)$ dimensions [16, 12]. It is defined by the relations

$$[x_1, x_0] = \frac{x_1}{\kappa} \quad [\alpha, x_0] = \frac{1}{\kappa} \sinh \alpha \quad [\alpha, x_1] = -\frac{1}{\kappa} (\cosh \alpha - 1) \quad (3.1)$$

(where $x_0 = ct$ (x_1) refers to time (space) and α characterizes the boost in the x direction) and by

$$\begin{aligned} \Delta \alpha &= \alpha \otimes 1 + 1 \otimes \alpha & \Delta x_0 &= 1 \otimes x_0 + x_0 \otimes \cosh \alpha - x_1 \otimes \sinh \alpha \\ \Delta x_1 &= 1 \otimes x_1 - x_0 \otimes \sinh \alpha + x_1 \otimes \cosh \alpha \\ S(\alpha) &= -\alpha & S(x_0) &= -x_0 \cosh \alpha - x_1 \sinh \alpha \\ S(x_1) &= -x_1 \cosh \alpha - x_0 \sinh \alpha & \epsilon(\alpha, x_0, x_1) &= 0. \end{aligned} \quad (3.2)$$

Introduce now the commutative, co-commutative Hopf algebra generated by a new generator $\hat{\chi}$ with dimensions of an action, such that

$$[\hat{\chi}, \text{all}] = 0 \quad \Delta \hat{\chi} = \hat{\chi} \otimes 1 + 1 \otimes \hat{\chi} \quad S(\hat{\chi}) = -\hat{\chi} \quad \epsilon(\hat{\chi}) = 0. \quad (3.3)$$

If we now make the redefinition $\chi = \hat{\chi} - mcx_0$ using the one-cochain mcx_0 (which diverges in the contraction limit) it follows that

$$\begin{aligned} \Delta \chi &= \chi \otimes 1 + 1 \otimes \chi - mc[x_0 \otimes (\cosh \alpha - 1) - x_1 \otimes \sinh \alpha] \\ S(\chi) &= -\chi + mc[x_0(\cosh \alpha - 1) + x_1 \sinh \alpha]. \end{aligned} \quad (3.4)$$

In terms of χ , the commutators in (3.3) imply

$$[\chi, x_0] = 0 \quad [\chi, x_1] = \frac{mc}{\kappa} x_1 \quad [\chi, \alpha] = \frac{mc}{\kappa} \sinh \alpha \quad (3.5)$$

and we see again that in order to have a sensible contraction limit we need to substitute $\hat{\kappa}c$ for κ , in which case equations (2.12) and (2.13) are recovered ($\alpha \sim v/c$) and hence $\text{Fun}_{\hat{\kappa}}(\tilde{G}_{(m)})$. Thus, there is full duality and closure among the algebra-like contraction diagrams for $\mathcal{U}_{\hat{\kappa}}(\tilde{G}_{(m)})$ [1] and the group-like ones for $\text{Fun}_{\hat{\kappa}}(\tilde{G}_{(m)})$.

A question that naturally arises is whether we can construct a similar extended group algebra from the Galilei deformation $\text{Fun}_{\hat{\kappa}}(G)$ dual of $\mathcal{U}_{\hat{\kappa}}(\mathcal{G})$, equations (1.1). As $\mathcal{U}_{\hat{\kappa}}(\mathcal{G})$, $\text{Fun}_{\hat{\kappa}}(G)$ has a bi-crossproduct structure so that the dualization (in two dimensions) of $\mathcal{U}(B) \bowtie \mathcal{U}_{\hat{\kappa}}(Tr_2)$ leads immediately to $\text{Fun}_{\hat{\kappa}}(G(1+1))$ as (cf (2.1))

$$\begin{aligned} [t, x] &= -\frac{1}{\hat{\kappa}} x & [x, v] &= \frac{v^2}{2\hat{\kappa}} & [t, v] &= -\frac{v}{\hat{\kappa}} \\ \Delta t &= t \otimes 1 + 1 \otimes t & \Delta x &= x \otimes 1 + 1 \otimes x - t \otimes v & \Delta v &= v \otimes 1 + 1 \otimes v \\ S(t, x, v) &= (-t, -x - vt, -v) & \epsilon(t, x, v) &= 0. \end{aligned} \quad (3.6)$$

Let us add a $\text{Fun}(U(1))$ factor to $\text{Fun}_{\hat{\kappa}}(G)$ (a similar discussion could be presented for $\mathcal{U}_{\hat{\kappa}}(\mathcal{G})$). Since we want the corresponding algebra generator to be central in $\mathcal{U}_{\hat{\kappa}}(\mathcal{G})$, α has to be trivial and hence its dual $\bar{\beta}$ is also trivial. The two-cocycle condition for the map $\bar{\psi} : A \rightarrow H \otimes H$ (here $A = \text{Fun}(U(1))$, $H = \text{Fun}_{\hat{\kappa}}(G)$) may be found as a consequence of the co-associativity requirement. It is given by [7]

$$\Delta \bar{\psi}(a_{(1)})^{(1)} \bar{\psi}(a_{(2)})^{(1)} \otimes \bar{\psi}(a_{(1)})^{(2)} a_{(2)}^{(2)} = \bar{\psi}(a_{(1)})^{(1)} \otimes \Delta \bar{\psi}(a_{(1)})^{(2)} \bar{\psi}(a_{(2)}) \quad (3.7)$$

where the lower indices refer to the coproduct and the upper ones refer either to $\bar{\beta}$ ($\bar{\beta}(a) = a^{(1)} \otimes a^{(2)} \in A \otimes H$) or to the components of $\bar{\psi}$ ($\bar{\psi}(a) = \bar{\psi}(a)^{(1)} \otimes \bar{\psi}(a)^{(2)}$). Since the right co-action $\bar{\beta}$ is trivial in our case, $a^{(1)} = a, a^{(2)} = 1$. Moreover, since

$\bar{\psi}(1) = 1_H \otimes 1_H$ and $\Delta a = a \otimes 1 + 1 \otimes a$, i.e. $\Delta \chi = \chi \otimes 1 + 1 \otimes \chi$, equation (3.7) reduces to

$$\Delta \bar{\psi}(\chi)^{(1)} \otimes \bar{\psi}(\chi)^{(2)} + \bar{\psi}(\chi) \otimes 1 = \bar{\psi}(\chi)^{(1)} \otimes \Delta \bar{\psi}(\chi)^{(2)} + 1 \otimes \bar{\psi}(\chi) \quad (3.8)$$

(it is easy to check, we note in passing, that (2.16) for $\text{Fun}_{\tilde{\kappa}}(\tilde{G}_{(m)})$ satisfies condition (3.8)). Thus, in the search for a $\text{Fun}_{\tilde{\kappa}}(G)$ we have to look for a $\bar{\psi}(\chi)$ which in the undeformed limit must reduce to the last term in (2.10). The usual dimension assignments (which we have consistently kept) indicate that a $\tilde{\kappa}$ -deformation of the Galilei two-cocycle (last bracket in (2.10)) may be described by

$$\bar{\psi}_{\tilde{\kappa}} = \bar{\psi}(\chi) \left(1 \otimes f \left(\frac{mv^2}{\hbar \tilde{\kappa}} \right) \right). \quad (3.9)$$

Omitting the constants we may now impose the two-cocycle condition (3.8)–(3.9) written as $\bar{\psi}_{\tilde{\kappa}} \propto x \otimes vf(v^2) - \frac{1}{2}t \otimes v^2 f(v^2)$. This leads to

$$\begin{aligned} \Delta x \otimes vf(v^2) - \frac{1}{2} \Delta t \otimes v^2 f(v^2) + x \otimes vf(v^2) \otimes 1 - \frac{1}{2} t \otimes v^2 f(v^2) \otimes 1 \\ = x \otimes \Delta(vf(v^2)) - \frac{1}{2} t \otimes \Delta(v^2 f(v^2)) + 1 \otimes x \otimes vf(v^2) \\ - \frac{1}{2} 1 \otimes t \otimes v^2 f(v^2) \end{aligned} \quad (3.10)$$

which implies

$$\begin{aligned} \Delta(vf(v^2)) &= 1 \otimes vf(v^2) + vf(v^2) \otimes 1 \\ \Delta(v^2 f(v^2)) &= 1 \otimes v^2 f(v^2) + v^2 f(v^2) \otimes 1 + 2v \otimes vf(v). \end{aligned} \quad (3.11)$$

These equations are inconsistent with the form of Δv in (3.6) and among each other unless f is constant. Thus, $\bar{\psi}$ is unmodified by $\tilde{\kappa}$ and $\Delta \chi$ is again given by (2.13). We may now try to complete the commutation relations in (3.6) with those for χ by imposing that the coproduct (as given by (3.6) and (2.13)) is an algebra homomorphism. Notice that the addition of a two-cocycle does not modify the expressions (3.6); only the commutators involving χ and $\Delta \chi$ need to be found. The result is that there is no solution in the presence of $\tilde{\kappa}$ if the constant f is non-zero†. This agrees with the fact that no $\tilde{\kappa}$ -deformation of the Hopf algebra $\mathcal{U}(\tilde{G}_{(m)})$ can be obtained from $\mathcal{U}_{\kappa}(\mathcal{P}) \times \mathcal{U}(u(1))$ by contraction [1].

4. Structure of the non-standard (1 + 1) deformed Poincaré group and their Galilean contractions

A non-standard deformation $\mathcal{U}_h(\mathcal{P})$ of the Poincaré Hopf algebra may be obtained by contraction from the $\mathcal{U}_h(sl(2, \mathbb{R}))$ deformation [18, 19] and has been recently studied [20–22]. We shall show first that it has a bi-crossproduct structure and then study its Galilean contractions.

In a ‘light-cone’ basis $\mathcal{U}_{\rho}(\mathcal{P}(1 + 1))$ (ρ is the parameter remaining after the contraction limit $\epsilon \rightarrow 0$ which is performed after setting $h = \epsilon\rho$, $[\epsilon] = L^{-1} = [\rho^{-1}]$) may be written in the form

$$\begin{aligned} [N, P_+] &= \frac{1 - \exp(-2\rho P_+)}{2\rho} & [N, P_-] &= -P_- & [P_+, P_-] &= 0 \\ \Delta P_+ &= P_+ \otimes 1 + 1 \otimes P_+ & \Delta P_- &= \exp(-2\rho P_+) \otimes P_- + P_- \otimes 1 \\ \Delta P_- &= \exp(-2\rho P_+) \otimes N + N \otimes 1 & S(N) &= -\exp(2\rho P_+)N \\ S(P_+) &= -P_+ & S(P_-) &= -\exp(2\rho P_+)P_- & \epsilon(N, P_{\pm}) &= 0. \end{aligned} \quad (4.1)$$

† This result disagrees with that in [17] in as much as the projective representations of $\text{Fun}_{\tilde{\kappa}}(G)$ are related to a two-cocycle Hopf group algebra extension (which is not discussed there).

$\mathcal{U}_\rho(\mathcal{P}(1+1))$ has a bi-crossproduct structure $\mathcal{K} = \mathcal{H} \bowtie \mathcal{A}$ where in this case \mathcal{H} is generated by N with primitive coproduct and \mathcal{A} is the Abelian Hopf subalgebra of $\mathcal{U}_\rho(\mathcal{P}(1+1))$ generated by the translations P_\pm , the right action $\alpha : \mathcal{A} \otimes \mathcal{H} \rightarrow \mathcal{A}$ and left coaction $\beta : \mathcal{H} \rightarrow \mathcal{A} \otimes \mathcal{H}$ being given, respectively, by

$$P_+ \triangleleft N = -\frac{1 - \exp(-2\rho P_+)}{2\rho} \quad P_- \triangleleft N = P_- \quad \beta(N) = \exp(-2\rho P_+) \otimes N. \quad (4.2)$$

We may easily see, for example, that the formulae which give the coproduct and antipode in \mathcal{K} [7],

$$\begin{aligned} \Delta_{\mathcal{K}}(h \otimes a) &= h_{(1)} \otimes h_{(2)}^{(1)} a_{(1)} \otimes h_{(2)}^{(2)} \otimes a_{(2)} \\ S(h \otimes a) &= (1_{\mathcal{H}} \otimes S_{\mathcal{A}}(h^{(1)} a))(S_{\mathcal{H}}(h^{(2)}) \otimes 1_{\mathcal{A}}) \end{aligned} \quad (4.3)$$

immediately reproduce ΔN and $S(N)$ (N is represented in \mathcal{K} by $N \otimes 1$ and, in equation (4.3), $N^{(1)} = \exp(-2\rho P_+)$, $N^{(2)} = N$ by (4.2) etc). It is simple to construct by duality ($\langle P_\pm, x_\pm \rangle = 1$) the associated non-standard $(1+1)$ spacetime Hopf algebra, which is defined by the relations

$$[x_+, x_-] = -2\rho x_- \quad \Delta x_\pm = x_\pm \otimes 1 + 1 \otimes x_\pm \quad S(x_\pm) = -x_\pm \quad \epsilon(x_\pm) = 0. \quad (4.4)$$

We are interested here, however, in constructing the whole dual Hopf algebra $\text{Fun}_\rho(G(1+1))$ and its possible extension, and in obtaining them from $\text{Fun}_\rho(P(1+1))$. Denoting the variable dual to N by α , $\langle N, \alpha \rangle = 1$, and using the fact that $\mathcal{U}_\rho(\mathcal{P}(1+1))$ is a bi-crossproduct, the duals $\bar{\beta}$ of α and $\bar{\alpha}$ ($\bar{\alpha}$) of β are found to be

$$\bar{\beta}(x_\pm) = x_\pm \otimes e^{\mp\alpha} \quad x_+ \bar{\alpha} = -2\rho(1 - e^{-\alpha}) \quad x_- \bar{\alpha} = 0 \quad (4.5)$$

(to find, e.g., $x_+ \bar{\alpha}$ one needs considering β for powers of N , $\beta(N^m)$). In this way the non-standard Poincaré Hopf algebra $\text{Fun}_\rho(P(1+1))$ is found to be

$$\begin{aligned} [x_+, \alpha] &= -2\rho(1 - e^{-\alpha}) & [x_-, \alpha] &= 0 & [x_+, x_-] &= -2\rho x_- \\ \Delta x_+ &= 1 \otimes x_+ + x_+ \otimes e^{-\alpha} & \Delta x_- &= 1 \otimes x_- + x_- \otimes e^\alpha \\ \Delta \alpha &= \alpha \otimes 1 + 1 \otimes \alpha & S(x_+) &= -x_+ e^\alpha \\ S(x_-) &= -x_- e^{-\alpha} & S(\alpha) &= -\alpha & \epsilon(x_\pm, \alpha) &= 0 \end{aligned} \quad (4.6)$$

which reproduces [22, 20].

We now find the non-standard deformed Galilei group $\text{Fun}_\rho(G(1+1))$. In terms of the standard $(x_0 = x_+ + x_-, x_1 = x_+ - x_-)$ basis, $\text{Fun}_\rho(P(1+1))$ is given by

$$\begin{aligned} [x_0, \alpha] &= -2\rho(1 - e^{-\alpha}) & [x_1, \alpha] &= -2\rho(1 - e^{-\alpha}) \\ [x_0, x_1] &= 2\rho(x_0 - x_1) \\ \Delta \alpha &= \alpha \otimes 1 + 1 \otimes \alpha & \Delta x_0 &= 1 \otimes x_0 + x_0 \otimes \cosh \alpha - x_1 \otimes \sinh \alpha \\ \Delta x_1 &= 1 \otimes x_1 - x_0 \otimes \sinh \alpha + x_1 \otimes \cosh \alpha \\ S(\alpha) &= -\alpha & S(x_0) &= -x_0 \cosh \alpha - x_1 \sinh \alpha \\ S(x_1) &= -x_1 \cosh \alpha - x_0 \sinh \alpha & \epsilon(\alpha, x_0, x_1) &= 0 \end{aligned} \quad (4.7)$$

which reproduces, in the co-algebra sector, the standard $P(1+1)$ group law. The non-standard deformation $\text{Fun}_\rho(G(1+1))$ now follows from the $c \rightarrow \infty$ limit for $x_0 = ct$, $x_1 = x$ and $\alpha \sim v/c$, with the result

$$\begin{aligned} \Delta t &= 1 \otimes t + t \otimes 1 & \Delta x &= 1 \otimes x + x \otimes 1 - t \otimes v & \Delta v &= 1 \otimes v + v \otimes 1 \\ [t, v] &= 0 & [x, v] &= -2\rho v & [t, x] &= 2\rho t \\ S(t) &= -t & S(x) &= -x - vt & S(v) &= -v. \end{aligned} \quad (4.8)$$

The dual algebra, $\mathcal{U}_\rho(\mathcal{G}(1+1))$, is easily found to be

$$\begin{aligned} [X_t, X] &= 0 & [X_t, V] &= -\frac{1}{4\rho}(1 - \exp(-4\rho X)) & [X, V] &= 0 \\ \Delta X_t &= X_t \otimes 1 + \exp(-2\rho X) \otimes X_t & \Delta X &= 1 \otimes X + X \otimes 1 & (4.9) \\ \Delta V &= V \otimes 1 + \exp(-2\rho X) \otimes V & S(X_t) &= -\exp(2\rho X)X_t \\ S(X) &= -X & S(V) &= -\exp(2\rho X)V & \epsilon(X_t, X, V) &= 0. \end{aligned}$$

It is not a new Hopf algebra; it is the deformed Heisenberg–Weyl algebra $\mathcal{U}_\rho(HW)$ (the quantum Heisenberg group $H_q(1)$ of [23]), and it has both a bi-crossproduct and a cocycle bi-crossproduct structure [5].

To complete the picture, it is interesting to close the contraction diagrams by obtaining $\mathcal{U}_\rho(\mathcal{G}(1+1))$ ($= \mathcal{U}_\rho(HW)$), equations (4.9), by contracting $\mathcal{U}_\rho(\mathcal{P}(1+1))$ in (4.1) by means of the standard redefinitions. It turns out, however, that the naïve change of basis ($P_0 = \frac{1}{2}(P_+ + P_-)$, $P_1 = \frac{1}{2}(P_+ - P_-)$) from light cone to standard variables is not adequate, and that a more complicated one (which may be justified [22] in terms of T matrix [24] considerations) is required to perform the contraction, namely, $P_+ = P_0 + P_1$, $P_- = (1/2\rho)[\exp(-2\rho(P_0 + P_1)) + 1 - 2\exp(-2\rho P_0)]$. In terms of these $P_{0,1}$ generators the algebra $\mathcal{U}_\rho(\mathcal{P}(1+1))$ is more complicated than in (4.1), but may be seen to lead to $\mathcal{U}_\rho(\mathcal{G}(1+1))$, equations (4.9), in the contraction limit.

Finally, we may look for a *non-standard extended Galilei group*. This can be obtained following the by now familiar procedure, which involves the addition of $\hat{\chi}$ as in (3.3), and the redefinition $\hat{\chi} = \chi + mcx_0$. Then (4.7) leads to (3.4) and to

$$[\chi, x_0] = 0 \quad [\chi, x_1] = -2mc\rho(x_0 - x_1) \quad [\chi, \alpha] = 2mc\rho(1 - e^{-\alpha}). \quad (4.10)$$

Equations (4.10) show that the contraction requires a redefinition of the deformation constant, $\rho = \hat{\rho}/c^2$, $[\hat{\rho}] = L^3 T^{-2}$. This leads to $\text{Fun}_{\hat{\rho}}(\tilde{\mathcal{G}}_{(m)}(1+1))$ defined by equations (2.1) and (2.13) together with

$$[\chi, t] = 0 \quad [\chi, x] = -2m\hat{\rho}t \quad [\chi, v] = 0. \quad (4.11)$$

$\text{Fun}_{\hat{\rho}}(\tilde{\mathcal{G}}_{(m)}(1+1))$ is a very mild deformation, only manifest in the $[\chi, x]$ commutator.

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